Semester One Examination, 2019

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 1

Section Two: Calculator-assumed

SOLUTIONS

Student number: In figures | | | | | | | | |

In words _____

Your name _____

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

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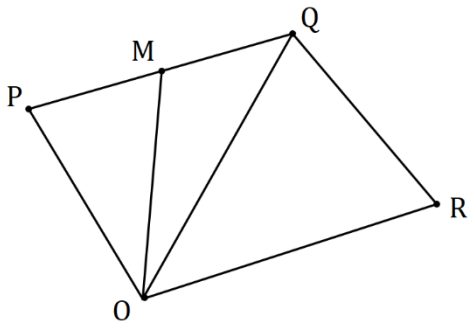
Section Two: Calculator-assumed **65% (98 Marks)**

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 9 **(8 marks)**

In the diagram below, M is the midpoint of PQ.



If $\overrightarrow{OP} = \mathbf{p}, \overrightarrow{OQ} = \mathbf{q}$ and $\overrightarrow{OR} = \mathbf{r}$, express the following in terms of $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$.

(a) $\overrightarrow{RP}$. (1 mark)

Solution
$\overrightarrow{RP} = \mathbf{p} - \mathbf{r}$
Specific behaviours
✓ correct expression

(b) $\overrightarrow{OM}$. (2 marks)

Solution
$\overrightarrow{OM} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ} = \mathbf{p} + \frac{1}{2}(\mathbf{q} - \mathbf{p})$ $= \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$
Specific behaviours
✓ determines $\overrightarrow{PQ}$ ✓ correct expression

(c) $4\overrightarrow{RM}$. (2 marks)

Solution
$\overrightarrow{RM} = \overrightarrow{RO} + \overrightarrow{OM} = -\mathbf{r} + \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$ $4\overrightarrow{RM} = 2\mathbf{p} + 2\mathbf{q} - 4\mathbf{r}$
Specific behaviours
✓ indicates $\overrightarrow{RM}$ ✓ correct expression

(d) Determine the value of λ and μ if $4\lambda\boldsymbol{p} + \boldsymbol{q} - 2\mu\boldsymbol{q} = -3\boldsymbol{p} - 3\mu\boldsymbol{p} + 3\lambda\boldsymbol{q} + 2\boldsymbol{q}$. (3 marks)

Solution	Specific behaviours
$(4\lambda + 3 + 3\mu)\boldsymbol{p} = (3\lambda + 1 + 2\mu)\boldsymbol{q}$ $\begin{cases} 4\lambda + 3 + 3\mu = 0 \\ 3\lambda + 1 + 2\mu = 0 \end{cases}$ Since $\boldsymbol{p}$ and $\boldsymbol{q}$ are non-parallel vectors $\lambda = 3, \ \mu = -5$	✓ collects vectors together ✓ forms simultaneous equations ✓ both solutions

See next page

Question 10 **(8 marks)**

(a) Show that the vectors $\begin{pmatrix}12\\-4\end{pmatrix}$ and $\begin{pmatrix}3\\9\end{pmatrix}$ are perpendicular. (2 marks)

Solution
$\begin{pmatrix}12\\-4\end{pmatrix}\cdot\begin{pmatrix}3\\9\end{pmatrix}=36-36=0$ Hence perpendicular as scalar (dot) product is 0.
Specific behaviours
✓ uses dot product ✓ explains result

(b) Determine, to the nearest degree, the angle between the vectors $\begin{pmatrix}-1\\3\end{pmatrix}$ and $\begin{pmatrix}-2\\2\end{pmatrix}$. (2 marks)

Solution
Using CAS: $\theta=26.56\approx27°$ Or: $\theta=\cos^{-1}\left(\frac{8}{\sqrt{10}\times\sqrt{8}}\right)$
Specific behaviours
✓ indicates method ✓ correct angle

(c) The vectors $\begin{pmatrix}a\\a-2\end{pmatrix}$ and $\begin{pmatrix}a-6\\4\end{pmatrix}$ are perpendicular, where a is a constant. Determine the value(s) of a and the corresponding pair(s) of vectors. (4 marks)

Solution
$\begin{pmatrix}a\\a-2\end{pmatrix}\cdot\begin{pmatrix}a-6\\4\end{pmatrix}=a^2-6a+4a-8=0$ $(a+2)(a-4)=0\Rightarrow a=-2,a=4$ $a=-2\Rightarrow\begin{pmatrix}-2\\-4\end{pmatrix}\text{ and }\begin{pmatrix}-8\\4\end{pmatrix}$ $a=4\Rightarrow\begin{pmatrix}4\\2\end{pmatrix}\text{ and }\begin{pmatrix}-2\\4\end{pmatrix}$
Specific behaviours
✓ uses dot product to form equation ✓ solves equation ✓ states one pair of vectors ✓ states both pairs of vectors

See next page

Question 11 **(8 marks)**

Points P, Q and R have position vectors $\begin{pmatrix}15\\-6\end{pmatrix}$, $\begin{pmatrix}-8\\7\end{pmatrix}$ and $\begin{pmatrix}16\\-3\end{pmatrix}$ respectively. Determine

(a) $\overrightarrow{PQ}$. (1 mark)

Solution
$\overrightarrow{PQ} = (-8,7) - (15,-6)$ $= (-23,13)$
Specific behaviours
✓ correct vector

(b) $|\overrightarrow{QR}|$. (2 marks)

Solution
$\overrightarrow{QR} = (16,-3) - (-8,7)$ $= (24,-10)$ $\left\|\overrightarrow{QR}\right\| = 26$
Specific behaviours
✓ correct vector ✓ magnitude

(c) $4\overrightarrow{PQ} + 52\mathbf{u}$, where $\mathbf{u}$ is a unit vector in the direction $\overrightarrow{QR}$. (3 marks)

Solution
$\mathbf{u} = \frac{1}{26}(24,-10)$ $4\overrightarrow{PQ} - 60\mathbf{u} = 4(-23,13) + \frac{52}{26}(24,-10)$ $= (-44,32)$
Specific behaviours
✓ indicates unit vector ✓ expression for result ✓ correct vector

(d) The position vector $\overrightarrow{OS}$ of point S, given that $\overrightarrow{SP} = \overrightarrow{RQ}$. (2 marks)

Solution
$\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS}$ $= \overrightarrow{OP} + \overrightarrow{QR}$ $= (15,-6) + (24,-10)$ $= (39,-16)$
Specific behaviours
✓ expression for result ✓ correct position vector

See next page

Question 12 **(8 marks)**

Determine how many of the integers between 1 and 500 inclusive are

(a) divisible by 7. (1 mark)

Solution
$\lfloor 500 \div 7 \rfloor = 71$ $n = 71$
Specific behaviours
✓ correct number

(b) divisible by 7 or 9. (3 marks)

Solution
LCM: $(7,9) = 63$; $\lfloor 500 \div 9 \rfloor = 55$ $\lfloor 500 \div 63 \rfloor = 7$ $n = 71 + 55 - 7 = 119$
Specific behaviours
✓ number divisible by 63 ✓ indicates use of inclusion-exclusion ✓ correct number

(c) divisible by 7 or 9 but not both. (1 mark)

Solution
$n = 119 - 7 = 112$
Specific behaviours
✓ correct number

(d) divisible by 7 or 9 but not 6. (3 marks)

Solution
LCM's: $(6,7) = 42$; $(6,9) = 18$; $(6,7,9) = 126$ $\lfloor 500 \div 18 \rfloor = 27$ $\lfloor 500 \div 42 \rfloor = 11$ $\lfloor 500 \div 126 \rfloor = 3$ $n = 119 - 27 - 11 + 3 = 84$
Specific behaviours
✓ divisible by $18, 42$ ✓ divisible by 126 ✓ correct number

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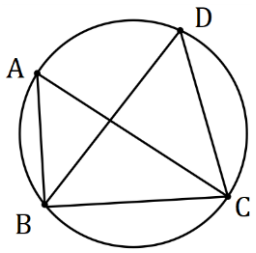
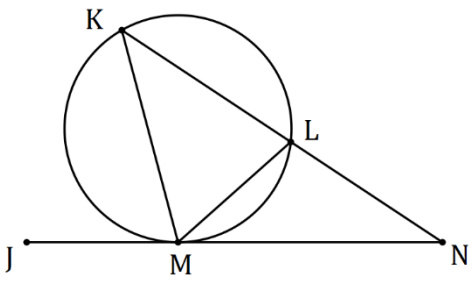
Question 13 **(8 marks)**

(a) A, B, C and D lie on a circle with diameter AC (diagram not to scale). Determine, with reasons, the size of $\angle BDC$ when $\angle BCA = 40°$. (4 marks)



Solution
$\angle ABC = 90°$ (angle in a semi-circle) $\angle BAC = 90° - 40° = 50°$ $\angle BDC = \angle BAC = 50°$ (angles in the same segment are equal)
Specific behaviours
✓ $\angle BAC = 90$ ✓ gives reason (angle in semi-circle) ✓ correct value of $\angle BDC$ ✓ gives reason

(b) K, L and M lie on a circle (diagram not to scale). Secant KN cuts the circle at L and JN is a tangent to the circle at M. Given that $\angle LNM = 33°$ and $\angle LMN = 43°$, determine the size of $\angle MKL$ and the size of $\angle KMJ$. Justify your answer. (4 marks)



Solution
$\angle MKL = \angle LMN = 43°$ (Alternate segments) $\angle KLM = 33 + 43 = 76$ (Exterior angle) $\angle KMJ = \angle KLM = 76°$ (Alternate segments)
Specific behaviours
✓ $\angle MKL$ ✓ $\angle KLM$ ✓ $\angle KMJ$ ✓ gives reasons for each angle

See next page

Question 14 **(9 marks)**

The parts of this question refer to the word BENEDICTIVE. It has 6 different consonants and 5 vowels, some of which are repeated.

(a) Determine the number of ways that 5 different consonants chosen from the letters of the word can be arranged in a row. (1 mark)

Solution
${}^6P_5 = 720$
Specific behaviours
✓ correct number

(b) Determine the number of ways that all the letters of the word can be arranged in a row. (2 marks)

Solution
$\dfrac{11!}{3! \times 2!} = 3\,326\,400$
Specific behaviours
✓ attempts to account for repeated letters ✓ correct number

(c) Determine the number of ways that all the letters of the word can be arranged in a row if the consonants must all be together. (3 marks)

Solution
$\dfrac{(1+5)! \times 6!}{3! \times 2!} = 43\,200$
Specific behaviours
✓ counts consonants as single group ✓ counts ways to arrange consonants ✓ correct number

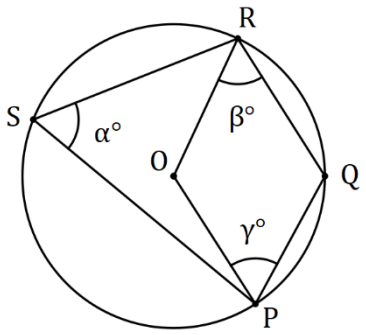
(d) Determine how many 3 letter permutations (e.g. TDI, EBE, etc) can be made using the letters of the word. (3 marks)

Solution
All different: $8 \times 7 \times 6 = 336$ Two I's and one other: $3 \times 7 = 21$ Two E's and one other: $3 \times 7 = 21$ Three E's: 1 Total: $336 + 21 + 21 + 1 = 379$
Specific behaviours
✓ attempts to consider separate cases ✓ correct number containing 2 I's and 2 E's ✓ correct total

See next page

Question 15 **(5 marks)**

In the diagram below (not drawn to scale) P, Q, R and S lie on the circle with centre O. Determine, with reasons, the size of angles α, β and γ given that $\angle PQR = 119°$ and $3\beta = 4\gamma$.



Solution
$\alpha = 180 - 119 = 61°$ (opposite angles in cyclic quadrilateral are supplementary) $\angle ROP = 2\alpha = 122°$ (angle at the centre is twice angle at the circumference) $\beta + \gamma + 119 + 122 = 360$ $\Rightarrow \beta + \gamma = 119$ (angle sum of quadrilateral) $\begin{cases} \beta + \gamma = 119 \\ 3\beta = 4\gamma \end{cases}$ $\Rightarrow \gamma = 51° \; and \; \beta = 68°$
Specific behaviours
✓ correct α with reason
✓ determines angle at the centre with reasons
✓ forms equation for $\beta + \gamma$
✓ correct γ
✓ correct β

See next page

Question 16 **(8 marks)**

(a) A set of cards is numbered with all the even numbers between 100 and 998. Determine the minimum number of cards that must be selected to ensure that at least 4 cards in the selection have the same last digit. Justify your answer using the pigeonhole principle.

(3 marks)

Solution
Let pigeonholes be digits $0, 2, 4, 6, 8$ and pigeons be the last digit of number on card. Then fill all pigeonholes with 3 pigeons, a total of 15 pigeons. The next pigeon will fill one of the pigeonholes with 4 pigeons, and so the minimum number is 16.
Specific behaviours
✓ defines pigeonholes ✓ clear explanation (e.g. worst case scenario) ✓ correct number (e.g. +1 to worst case)

(b) Seven different books sit on a shelf, one of which is fiction and the rest non-fiction. A student is told they can take away as many of them as they like but must not leave empty handed. Determine how many different selections can be made

(i) of exactly 4 books. (1 mark)

Solution
$\binom{7}{4} = 35$
Specific behaviours
✓ correct number

(ii) altogether. (2 marks)

Solution
Choose either $1, 2, \ldots$ up to all 7 books: $\sum_{n=1}^{7} \binom{7}{n} = 2^7 - 1 = 127$
Specific behaviours
✓ indicates method ✓ correct number

(iii) that include the fiction. (2 marks)

Solution
Choose fiction and then $0, 1, \ldots$ up to 6 others: $\binom{1}{1} \times \sum_{n=0}^{6} \binom{6}{n} = 2^6 = 64$
Specific behaviours
✓ indicates method ✓ correct number

See next page

Question 17 **(8 marks)**

A and B have position vectors $5\mathbf{i} - 10\mathbf{j}$ and $-11\mathbf{i} + 15\mathbf{j}$ respectively.

Particle P is initially at A and moves with a constant velocity of $12\mathbf{i} - 5\mathbf{j}$ ms^{-1}.

(a) Calculate

(i) the speed of P. (1 mark)

Solution
$s = \sqrt{(12)^2 + (-5)^2} = 13 \text{ m/s}$
Specific behaviours
✓ correct speed

(ii) the position vector of P after 2 seconds. (1 mark)

Solution
$\begin{pmatrix} 5 \\ -10 \end{pmatrix} + 2\begin{pmatrix} 12 \\ -5 \end{pmatrix} = \begin{pmatrix} 29 \\ -20 \end{pmatrix}$
Specific behaviours
✓ correct position

(iii) the distance of P from B after 2 seconds. (2 marks)

Solution
$\overrightarrow{PB} = \begin{pmatrix} -11 \\ 15 \end{pmatrix} - \begin{pmatrix} 29 \\ -20 \end{pmatrix} = \begin{pmatrix} -40 \\ 35 \end{pmatrix}$
$\left\|\overrightarrow{PB}\right\| = \sqrt{(-40)^2 + (35)^2} = 5\sqrt{113} \approx 53.2 \text{ m}$
Specific behaviours
✓ vector $\overrightarrow{PB}$
✓ correct distance

(b) Determine how long after leaving A that P is 233 m from B. (4 marks)

Solution
$\overrightarrow{OP} = \begin{pmatrix} 5 \\ -10 \end{pmatrix} + t\begin{pmatrix} 12 \\ -5 \end{pmatrix}$
$\overrightarrow{PB} = \begin{pmatrix} -11 \\ 15 \end{pmatrix} - \begin{pmatrix} 5 + 12t \\ -10 - 5t \end{pmatrix} = \begin{pmatrix} -16 - 12t \\ 25 + 5t \end{pmatrix}$
$\lvert\overrightarrow{PB}\rvert^2 = (-16 - 12t)^2 + (25 + 5t)^2 = 233^2$
$t = 16$
i.e. after 16 seconds
Specific behaviours
✓ expression for $\overrightarrow{OP}$
✓ expression for $\overrightarrow{PB}$
✓ equation using distance
✓ correct time

See next page

Question 18 **(7 marks)**

Three forces $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ act on a point in a plane.

The forces are $\mathbf{a} = -53\mathbf{i} + 19\mathbf{j}$ N, $\mathbf{b} = -25\mathbf{i} - 48\mathbf{j}$ N and $\mathbf{c} = 309\mathbf{i} + 231\mathbf{j}$ N.

(a) Determine the magnitude of the resultant force and the direction, to the nearest degree, that the resultant makes with the horizontal unit vector $\mathbf{i}$. (3 marks)

Solution
$\mathbf{r} = \begin{pmatrix} -53 \\ 19 \end{pmatrix} + \begin{pmatrix} -25 \\ -48 \end{pmatrix} + \begin{pmatrix} 309 \\ 231 \end{pmatrix} = \begin{pmatrix} 231 \\ 202 \end{pmatrix}$ $\lvert\mathbf{r}\rvert = 306.9 \approx 307$ N $\angle = 41.2 \approx 41°$
Specific behaviours
✓ resultant ✓ correct magnitude ✓ correct angle (allow any reasonable rounding for magnitude and angle)

When $\lambda\mathbf{a} + \mu\mathbf{b} + \mathbf{c} = 0$, the forces are in equilibrium.

(b) Determine the values of the scalar constants λ and μ for equilibrium to occur. (4 marks)

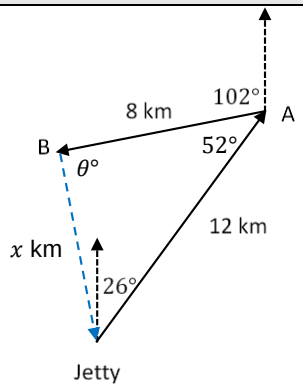
Solution
$\lambda\begin{pmatrix} -53 \\ 19 \end{pmatrix} + \mu\begin{pmatrix} -25 \\ -48 \end{pmatrix} + \begin{pmatrix} 309 \\ 231 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{cases} -53\lambda - 25\mu + 309 = 0 \\ 19\lambda - 48\mu + 231 = 0 \end{cases}$ $\lambda = 3, \quad \mu = 6$
Specific behaviours
✓ equation using $\mathbf{i}$-coefficients ✓ equation using $\mathbf{j}$-coefficients ✓ solves for λ ✓ solves for μ

Question 19 **(5 marks)**

Lay is planning a route for his sailing practice involving three successive straight legs.

The first leg from the jetty to point A is 12 km at a bearing of $026°T$. The second leg from point A to point B is 8 km at a bearing of $258°T$. The final leg needs to bring him back to the jetty from point B.

(a) Draw a clearly labelled diagram to represent his route. (1 mark)

Solution	Specific behaviours
	✓ correct diagram with labelled points, lengths (8 km and 12 km) and angles/bearings ($26°$ and $102°$ or $52°$)

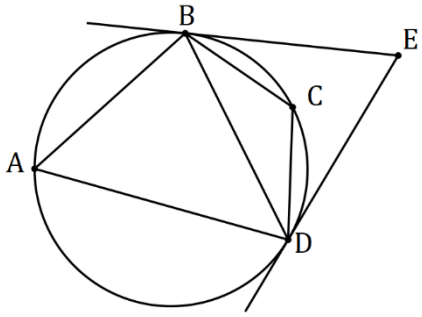
(b) Determine the required distance (to 2 decimal places) and bearing (to the nearest degree) for the final leg of the route. (4 marks)

Solution
$x^2 = 8^2 + 12^2 - 2 \times 8 \times 12 \times \cos 52$ $x = 9.48$ km $\cos\theta = \dfrac{8^2 + x^2 - 12^2}{2 \times 8 \times x}$ $\theta = 86.3°$ Bearing $= \theta + 78° = 164.3°$ Hence final leg is 9.48 km at a bearing of $164°T$.
Specific behaviours
✓ determines distance ✓ determines an angle in the triangle ✓ converts to bearing ✓ correct rounding

See next page

Question 20 **(9 marks)**

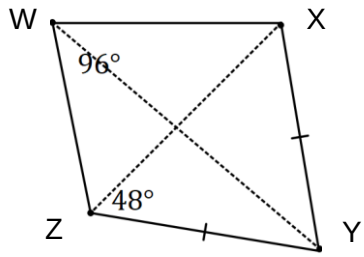
(a) In the diagram below (not drawn to scale) A, B, C and D lie on a circle and EB and ED are tangents to the circle. If $\angle BED = 54°$ and $\angle CDB = 20°$, determine the size of $\angle CBD$. Justify your answer. (4 marks)



Solution
$BE = DE$ (two tangents to a circle are equal length) Hence ΔBDE is isosceles $\angle BDE = (180° - 54°) \div 2 = 63°$ (base angles equal in isosceles) $\angle CDE = 63° - 20° = 43°$ $\angle CBD = \angle CDE = 43°$ (angle between tangent and chord equals angle in the alternate segment)
Specific behaviours
✓ $BE = DE$ with reason
✓ $\angle BDE$ with reason
✓ $\angle CDE$
✓ $\angle CBD$ with reason

(b) Quadrilateral $WXYZ$ is such that $YX = YZ$, $\angle XWZ = 96°$ and $\angle XZY = 48°$.

(i) Sketch a diagram to show this information. (1 mark)

Solution

Specific behaviours
✓ correct diagram

(ii) Show that $WXYZ$ is cyclic and hence determine, with reasons, the size of $\angle YWZ$. (4 marks)

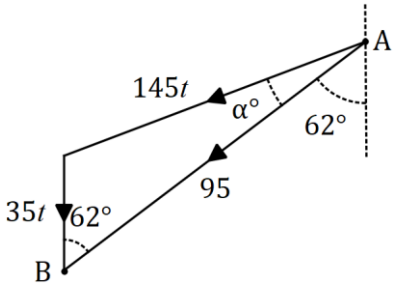
Solution
$\angle YXZ = \angle YZX = 48$ (base angles of isosceles equal) $\angle XYZ = 180 - 2 \times 48 = 84$ (angle sum of triangle) $\angle XWZ + \angle XYZ = 96 + 84 = 180$ Hence, $WXYZ$ is cyclic since its opposite angles are supplementary $\angle YWZ = \angle YXZ = 48°$ (angles in the same segment are equal)
Specific behaviours
✓ use isosceles triangle for $\angle XYZ$
✓ uses supplementary angles for cyclic
✓ correct size of $\angle YWZ$
✓ reason for $\angle YWZ$

See next page

Question 21 **(7 marks)**

Farm A lies 95 km away from farm B on a bearing of $062°$. A helicopter leaves farm A at $7:30$ am to fly to farm B. The helicopter can maintain a speed of 145 kmh^{-1} and there is a steady wind of 35 kmh^{-1} blowing from the north.

Determine the bearing that the helicopter should steer and the time of its arrival at farm B, to the nearest minute.

Solution	Alternative Trigonometric Solution
 $\frac{\sin 62}{145t} = \frac{\sin \alpha}{35t}$ $\alpha = 12.3°$ Bearing: $180 + 62 + 12.3 = 254.3°$ $180 - 62 - 12.3 = 105.7$ $\frac{\sin 62}{145t} = \frac{\sin 105.7}{95}$ $t = 0.601$ h $= 36$ m Arrive at $8:06$ am	*Diagram* *Similar to that on the left,* *but without the time factor included* Resultant speed = x km/h $145^2 = 35^2 + x^2 - 2 \times x \times 35 \times \cos 62°$ CAS solve: $x = 158.1$ km/h ($x > 0$) Time taken: $\frac{95}{158.1} = 0.601\ hrs = 36\ min$ ∴ Time of arrival 8:06 am [*Sine rule for angle similar to that on the left*] $35^2 = 145^2 + 158.1^2 - 2 \times 145 \times 158.1 \times \cos \alpha$ $\alpha = 12.3°$ Bearing: $180 + 62 + 12.3 = 254.3°$ Hence helicopter should set out at a bearing of $254.3°T$ and will arrive at farm B at 8:06 am.
Specific behaviours	**Specific behaviours**
✓ diagram showing vectors and resultant ✓ equation using sin rule for α ✓ value of α ✓ correct bearing ✓ equation using sin rule for t ✓ value of t ✓ correct arrival time	✓ diagram showing vectors and resultant ✓ equation using cos rule for resultant speed ✓ value of speed or calculates time taken ✓ correct time of arrival ✓ equation using sin or cos rule for α ✓ value of α ✓ correct bearing

End of questions

Supplementary page

Question number: _____

Supplementary page

Question number: _____

Supplementary page

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Supplementary page

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